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DIFFRACTION OF A WAVE AT HIGH FREQUENCIES BY AN ARBITRARY SMOOTH CONVEX OBJECT

by

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SUMMARY

In this paper the three dimensional scattering of a plane and a spherical wave by an arbitrary smooth convex object will be considered.

These problems are solved for large values of frequency by means of ray theory and the theory of boundary layer expansions.

1. *Introduction.*

The problem to be treated is the three-dimensional scattering of a scalar wave, by a totally reflecting smooth object, at high frequencies. The medium we consider is homogeneous. In this article we apply the ray method of J. B. Keller [1-3] to solve this problem. This method based on the ray-concept is combined with the method of boundary-layer expansions. The reasoning being essentially physical in nature makes these methods more advantageous in comparison with other methods. Although exact proofs are not available, the correctness of this theory is not seriously questioned and it is believed that it yields asymptotic representations, valid for short wavelength, of the exact solutions of the Helmholtz equation. The evidence for this comes from exact solutions, valid for a few simple geometrical configurations, e.g. the two dimensional circular cylinder and the sphere, for which short-wave expansions can be found rigorously. V. A. Fock [4] in his two dimensional treatment of short-wave diffraction by a convex cylinder used the ray method and boundary layer expansions as well.

As mentioned before we are interested in solutions of the Helmholtz equation

$$\Delta \tilde{\varphi} + k^2 \tilde{\varphi} = 0 \tag{1,1}$$

with $(k\ a) \gg 1$ where a is a measure for the radius of curvature of the object. It seems that in our treatment we must have $(k\ a)^{1/3} \gg 1$. The boundary condition on the object is $\tilde{\varphi} = 0$. The wave function $\tilde{\varphi}$ can be interpreted either as the velocity potential of sound waves, corresponding to an acoustically soft object, or as the Schrödinger wave function in non-relativistic quantum mechanics [5] in which case it corresponds to a hard-core potential. There is no difficulty in extending the treatment to a vector wave field, so as to represent electromagnetic scattering from a perfectly conducting object in a homogeneous medium.

To apply the "ray" method we have to define rays first. Rays are introduced by means of the substitution

$$\tilde{\varphi}(\underline{x}, k) = \varphi(\underline{x}, k) e^{ikS(\underline{x})} \tag{1,2}$$

in which $\underline{x} = (x, y, z)$.

Inserting (1,2) in (1,1) we obtain the equation

$$\Delta\varphi + ik(2\nabla\varphi \cdot \nabla S + \varphi\Delta S) + k^2\left[1-(\nabla S)^2\right]\varphi = 0 \tag{1,3}$$

If φ and S do not have large gradients, then for large values of k the k^2 term is the leading one. Hence we put its coefficient equal to zero and we get

$$(\nabla S)^2 = 1. \tag{1,4}$$

In geometrical optics this equation is the well-known eiconal equation. Its characteristics are straight lines which are perpendicular to surfaces S = constant. If we introduce the arclength ξ along these characteristics and choose ξ positively in the direction of increasing S, then we get along these lines

$$S = S_0 + \xi \tag{1,5}$$

where S_0 is an integration constant. We call these characteristics rays.

In geometrical optics rays are often introduced by means of the Fermat principle. In general we can easily verify [8] that in a nonhomogeneous medium the characteristics of the equation

$$(\nabla S)^2 = n^2(\underline{x})$$

in which $n(\underline{x})$ is the refractive index, (we arrive at this eiconal equation, if we put in equation (1.1) $k^2 n^2(\underline{x})$ instead of k^2) are the curves between two fixed points $\underline{x}_1$ and $\underline{x}_2$ along which the optical path length

$$\int_C n(\underline{x}) ds$$

is stationary with respect to small variations in the integration path C.

The case in question deals with a homogeneous medium with $n(\underline{x}) = 1$, so the rays are straight lines. The incoming rays of our problem are spherical or parallel to the x-axis with unit amplitude. If we put the object in this field, we find two different regions, viz. a lit region and a shadow region.

In the lit region every point is reached by two rays, a direct incoming ray and a ray reflected by the object. In this way no rays will come into the shadow region. Hence we are dealing with a solution of (1,1) which is discontinuous along the shadow boundary and the assumption that the gradients are moderate is violated, so the k^2 term is not the leading one anymore. If we still require solutions of the form (1,2), we must remove the discontinuity. We therefore define creeping rays.

Creeping rays are rays generated by that part of the incoming rays which meets the surface tangentially and follows the geodesics of the object in a direction which is the same as the incoming rays. Each point of these geodesics will generate a ray into free space tangentially to the geodesic line with a certain amplitude.

The geometrical considerations yield the solution

$$\tilde{\varphi} = \tilde{\varphi}_{inc} + \tilde{\varphi}_{refl} + \tilde{\tilde{\varphi}}_d \tag{1,6}$$

in the lit region and

$$\tilde{\varphi} = \tilde{\varphi}_d \tag{1,7}$$

in the shadow region, where $\tilde{\varphi}_{inc}$ and $\tilde{\varphi}_{refl}$ are the incident and the reflected ray, respectively. $\tilde{\tilde{\varphi}}_d$ and $\tilde{\varphi}_d$ are the diffracted rays in the lit and the shadow region, respectively. These diffracted rays are caused by the creeping rays. We will see that for large values of k the influence of the diffracted rays in the lit region far from the shadow boundary is asymptotically small with respect to the incident and the reflected ray.

In the next section we meet solutions of the form

$$\tilde{\varphi}(\underline{x}, k) = \tilde{\varphi}(\underline{x}, k)\exp\left\{ik\,S(\underline{x}) - k^s\psi(\underline{x})\right\}$$

with $s = \frac{1}{3}$ instead of the form (1, 2). It can be proved that these solutions are asymptotic solutions of (1, 1) also [6].

2. *Derivation of the asymptotic equations.*

We investigate the three dimensional diffraction of waves in a homogeneous medium by a smooth convex object. Two cases are considered, namely a plane and a spherical incident wave. Although the object is the same we must introduce a different coordinate system in both cases. We only discuss the field caused by the creeping rays in the shadow region. In section 1 we defined creeping rays as geodesics starting on the shadow boundary in the direction of the incident rays. We call these lines u^2 = constant. The arclength along these rays is defined by u^1 and the shadow boundary is $u^1 = h(u^2)$. The coordinate lines on the surface, u^1 = constant, are perpendicular to the geodesics. The surface is determined by the vector

$$x^i = x^i(u^\alpha) \qquad \begin{matrix} i = 1, 2, 3 \\ \alpha = 1, 2 \end{matrix} \tag{2, 1}$$

and the linear element on the surface

$$ds^2 = {du^1}^2 + g\,{du^2}^2 \tag{2, 2}$$

with

$$g_{ij} = \delta_{kl}\frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}, \quad g_{11} = 1, \quad g_{12} = 0 \text{ and } g_{22} = g$$

where

$$\delta_{kl} = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{cases}$$

The diffracted rays are tangentially to the geodesics and therefore we introduce the new coordinates u^1, u^2 and u^3 as follows

$$z^i(u^1, u^2, u^3) = x^i(u^1, u^2) + (u^3 - u^1)\frac{\partial x^i(u^1, u^2)}{\partial u^1} \tag{2, 3}$$

with $i = 1, 2, 3$ and $u^3 - u^1 \geqq 0$

where z^i are the Cartesian coordinates of a point in space. In the u^i coordinates we derive the Laplace operator Δ which has the form

$$\Delta = G^{ij}\left\{\frac{\partial^2}{\partial u^i \partial u^j} - \Gamma^k_{ij}\frac{\partial}{\partial u^k}\right\} \tag{2, 4}$$

G^{ij} is the contravariant matric tensor and Γ^k_{ij} the Christoffel symbol of the second kind.

First we derive the covariant metric tensor

$$G_{ij} = \delta_{kl} \frac{\partial z^k}{\partial u^i} \frac{\partial z^l}{\partial u^j}$$

we find

$$G_{11} = (u^3 - u^1)^2 \delta_{ij} \frac{\partial^2 x^i}{\partial u^{1^2}} \frac{\partial^2 x^j}{\partial u^{1^2}}$$

$$G_{12} = (u^3 - u^1)^2 \delta_{ij} \frac{\partial^2 x^i}{\partial u^{1^2}} \frac{\partial^2 x^j}{\partial u^1 \partial u^2} \tag{2, 5}$$

$$G_{22} = g + (u^3 - u^1) \frac{\partial g}{\partial u^1} + (u^3 - u^1)^2 \delta_{ij} \frac{\partial^2 x^i}{\partial u^1 \partial u^2} \frac{\partial^2 x^j}{\partial u^1 \partial u^2}$$

$$G_{13} = G_{23} = 0$$

$$G_{33} = 1$$

From (2, 2) follows that the unit vector n_i normal to the surface has the form

$$\frac{\partial^2 x^i}{\partial u^{1^2}} = \frac{1}{\rho(u^1, u^2)} n_i \tag{2, 6}$$

with $\rho(u^1, u^2)$ as the radius of curvature of the geodesic u^2 = constant.
The determinant G of the metric tensor is

$$G = \frac{(u^3 - u^1)^2}{\rho^2} \left\{ g + (u^3 - u^1) \frac{\partial g}{\partial u^1} \right\} +$$

$$+ (u^3 - u^1)^4 \left\{ \frac{1}{\rho^2} \delta_{ij} \frac{\partial^2 x^i}{\partial u^1 \partial u^2} \frac{\partial^2 x^j}{\partial u^1 \partial u^2} - \left(\delta_{ij} \frac{\partial^2 x^i}{\partial u^{1^2}} \frac{\partial^2 x^j}{\partial u^1 \partial u^2} \right)^2 \right\}$$

and with help of (2, 2) we find

$$G = \frac{(u^3 - u^1)^2}{\rho^2} \left\{ g^{\frac{1}{2}} + (u^3 - u^1) \frac{\partial g^{\frac{1}{2}}}{\partial u^1} \right\}^2 \tag{2, 7}$$

We introduce

$$R = \delta_{ij} \frac{\partial^2 x^i}{\partial u^{1^2}} \frac{\partial^2 x^j}{\partial u^1 \partial u^2}$$

And find

$$G^{11} = \frac{1}{G} \left[\left\{ g^{\frac{1}{2}} + (u^3 - u^1) \frac{\partial g^{\frac{1}{2}}}{\partial u^1} \right\}^2 + (u^3 - u^1)^2 \rho^2 R^2 \right]$$

$$G^{12} = - \frac{1}{G} (u^3 - u^1)^2 R$$

$$G^{22} = \frac{1}{G} \frac{(u^3 - u^1)^2}{\rho^2} \tag{2, 8}$$

$$G^{13} = G^{23} = 0$$

$$G^{33} = 1$$

In section 1 we noticed, the wave function is of the form

$$\widetilde{\varphi}_d = \varphi_d \exp\{ik(S_o + u^3)\}$$

with S_0 being constant.

Introducing this substitution here, we obtain the following form of the Helmholtz equation

$$G^{ij}\left(\frac{\partial^2\varphi_d}{\partial u^i \partial u^j} - \Gamma^k_{ij}\frac{\partial\varphi_d}{\partial u^k}\right) + ik\left\{2\frac{\partial\varphi_d}{\partial u^3} + \frac{\varphi_d}{u^3 - u^1} + \frac{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}\varphi_d}{g^{\frac{1}{2}} + (u^3 - u^1)\frac{\partial g^{\frac{1}{2}}}{\partial u^1}}\right\} = 0 \qquad (2,9)$$

On the object the condition $\varphi_d = 0$ must be satisfied, at infinity we require the radiation condition and on the shadow boundary, at finite distance from the object

$$\widetilde{\varphi}_d \approx \widetilde{\varphi}_{inc}$$

An asymptotic solution of (2,9) can be found by equating the term with the highest power of k to zero. This leads to an asymptotic solution of the form

$$\varphi_d = \frac{F(u^1, u^2)}{\left[(u^3 - u^1)\left\{g^{\frac{1}{2}} + (u^3 - u^1)\frac{\partial g^{\frac{1}{2}}}{\partial u^1}\right\}\right]^{\frac{1}{2}}} \qquad (2,10)$$

The function $F(u^1, u^2)$ is an integration constant of the asymptotic equation. Solution (2,10) is singular on the object where $u^3 = u^1$ and on the caustic surface

$$g^{\frac{1}{2}} + (u^3 - u^1)\frac{\partial g^{\frac{1}{2}}}{\partial u^1} = 0$$

In the case of diffraction by a sphere this caustic is reduced to the axis of symmetry [7]. With our method it is, in principle, possible to give asymptotic solutions near the caustic as well. In this article we consider solutions valid at finite distance from the caustic if any.

We now apply the method of boundary-layer expansions to find a solution, valid near the object. We therefore stretch the coordinates near the object and the shadow boundary. The shadow boundary on the object is the given line $u^1 = h(u^2)$. We introduce the new coordinates

$$\alpha = k^{\frac{1}{3}}\{u^3 - h(u^2)\}$$

$$\beta = k^{\frac{1}{3}}\{u^1 - h(u^2)\}$$

Putting this in equation (2,9) and equating the coefficient of the highest power of k to zero we get a differential equation for the asymptotic solution

$$\frac{\rho^2}{\alpha-\beta}\frac{\partial}{\partial\beta}\frac{1}{\alpha-\beta}\frac{\partial\varphi_d}{\partial\beta} + 2i\frac{\partial\varphi_d}{\partial\alpha} + \frac{i\varphi_d}{\alpha-\beta} = 0 \qquad (2,11)$$

where we take for ρ the value of the radius of curvature on the shadow boundary on the surface $\rho = \rho\{h(u^2), u^2\}$.
Introducing the coordinates

$$p = 2^{-\frac{2}{3}} \rho^{-\frac{4}{3}}(\alpha - \beta)^2$$

$$q = 2^{-\frac{1}{3}} \rho^{-\frac{2}{3}} \alpha$$

and the new function χ

$$\chi = \varphi_d \exp(i^{\frac{2}{3}} p^{\frac{3}{2}})$$

We arrive at the equation for χ

$$\frac{\partial^2\chi}{\partial p^2} + i\frac{\partial\chi}{\partial q} + p\chi = 0 \qquad (2,12)$$

with boundary condition $\chi = 0$ if $p = 0$.
The solution of this equation depends on $\widetilde{\varphi}_{inc}$. We therefore consider different forms of the incident wave.

3. *The incident wave is plane.*

Dealing with a plane incident wave we obtain a solution (2,12) of the form

$$\chi = A \cdot \int_C e^{itq} \left\{ W_2(t-p) - \frac{W_2(t)}{W_1(t)} W_1(t-p) \right\} dt \qquad (3,1)$$

where $W_1(t)$ and $W_2(t)$ are Airy functions

$$W_1(t) = e^{\frac{2}{3}\pi i} \sqrt{\frac{\pi}{3}} (-t)^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}} \left\{ \frac{2}{3}(-t)^{\frac{3}{2}} \right\}$$

$H^{(1)}_{\frac{1}{3}}$ is a Hankel function of the first kind and order. The function $W_2(t)$ is the complex conjugate function of $W_1(t)$. We have

$$W_1(t) = u(t) + i\, v(t)$$

$$W_2(t) = u(t) - i\, v(t)$$

The contour C runs from $\infty\, e^{\frac{2}{3}\pi i}$ over 0 to ∞. This contour can be closed at infinity in the half plane where the imaginary part of t is positive. The integral along this closing integral is equal to zero. The zeros of $W_1(t)$ are points on the line $t = \rho\, e^{\frac{1}{3}\pi i}$ for real positive ρ.

To find the constant A we use the condition on the shadow boundary, which states that at finite distance from the object the asymptotic solution tends to φ_{inc}.
Therefore we continue (3,1) in that region with the help of (2,10) and obtain the solution

$$\widetilde{\varphi}_d \approx \frac{A \exp\left\{-\frac{2}{3} p^{\frac{3}{2}} + ik(S_0 + u^3)\right\}}{\left\{ g^{\frac{1}{2}} + (u^3 - u^1) \frac{\partial g^{\frac{1}{2}}}{\partial u^1} \right\}^{\frac{1}{2}}} \cdot \int_C e^{itq} \left\{ W_2(t-p) - \frac{W_2(t)}{W_1(t)} W_1(t-p) \right\} dt$$

Referring to Eisenhart [9] we know that the principal radius of curvature of the wave fronts of the incoming wave which are tangentially to the object on the shadow boundary

$$\rho_1 = \frac{g^{\frac{1}{2}}\left\{h(u^2), u^2\right\}}{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}}$$

In this case, we are dealing with a plane incident wave, hence $\rho_1 = \infty$ this leads to $\frac{\partial g^{\frac{1}{2}}}{\partial u^1} = 0$ on the shadow boundary

For large values of p we get

$$\tilde{\varphi}_{inc} \approx \frac{4\sqrt{\pi}}{i} A \frac{\exp\left\{ik(S_o+u^3)\right\}}{g^{\frac{1}{4}}}$$

We determine S_o in such a way that

$$\tilde{\varphi}_{inc} = \exp\left\{ik(S_o+u^3)\right\}.$$

This can always be done.

Hence we have

$$A = \frac{i}{4\sqrt{\pi}} g$$

and the asymptotic solution near the shadow boundary is

$$\tilde{\varphi}_d \approx \frac{i}{4\sqrt{\pi}} \frac{g^{\frac{1}{4}} \exp\left\{ik(S_o+u^3) - i\frac{2}{3}p^{\frac{3}{2}}\right\}}{\left\{g^{\frac{1}{2}} + (u^3-u^1)\frac{\partial g^{\frac{1}{2}}}{\partial u^1}\right\}^{\frac{1}{2}}} \times \tag{3,2}$$

$$\times \int_C e^{itq} \left\{W_2(t-p) - \frac{W_2(t)}{W_1(t)} W_1(t-p)\right\} dt$$

where

$$p = \left(\frac{k}{2}\right)^{\frac{2}{3}} \rho^{-\frac{4}{3}} \left\{h(u^2),\ u^2\right\}(u^3-u^1)^2$$

$$q = \left(\frac{k}{2}\right)^{\frac{1}{3}} \rho^{-\frac{2}{3}} \left\{h(u^2),\ u^2\right\}\left[(u^3-h(u^2))\right]$$

Writing (3,2) as a sum of residues we find

$$\tilde{\varphi}_d \approx \frac{\sqrt{\pi}\ g^{\frac{1}{4}} \exp\left\{ik(S_o+u^3) - i\frac{2}{3}p^{\frac{3}{2}}\right\}}{\left\{g^{\frac{1}{2}} + (u^3-u^1\frac{\partial g^{\frac{1}{2}}}{\partial u^1}\right\}^{\frac{1}{2}}} \sum_{s=0}^{\infty} e^{it_s q} \frac{W_1(t_s-p)}{\left\{W_1{}^1(t_s)\right\}},$$

bearing in mind that $W_1(t_s) = 0$

At finite distance from the object we expand the Airy function for large values of p and we get

$$\widetilde{\varphi}_d \approx \frac{\left(\frac{k}{2}\right)^{-\frac{1}{6}} \sqrt{\pi}\ \rho^{\frac{1}{3}}\left\{h(u^2), u^2\right\}\ g^{\frac{1}{4}} \exp\left\{ik(S_0+u^3)+\frac{\pi i}{4}\right\}}{\left[(u^3-u^1)\left\{g^{\frac{1}{2}}+(u^3-u^1)\dfrac{\partial g^{\frac{1}{2}}}{\partial u^1}\right\}\right]^{\frac{1}{2}}} \times$$

$$\times \sum_{s=0}^{\infty} \frac{\exp\left\{it_s\left(\frac{k}{2}\right)^{\frac{1}{3}}\rho^{-\frac{3}{2}}\left\{h(u^2), u^2\right\}\ \left[u^1-h(u^2)\right]\right\}}{\left\{W_1{}^1(t_s)\right\}^2} \tag{3, 3}$$

This solution is valid near the shadow boundary. Assuming that at finite distance from this shadow the exponential behaviour of the solution is

$$\exp\left\{ik(S_0+u^3) + it_s\left(\frac{k}{2}\right)^{\frac{1}{3}} \int_{h(u^2)}^{u^1} \rho^{-\frac{2}{3}}(u^1, u^2)du^1\right\},$$

we are able to derive a solution in that region. Introducing the variable

$$\gamma = \rho^{-\frac{2}{3}}\ k^{\frac{1}{3}}\ (u^3-u^1)$$

and the solution of the form

$$\widetilde{\varphi}_d = \sum_{s=0}^{\infty} \exp\left\{ik(S_0+u^3)+it_s\left(\frac{k}{2}\right)^{\frac{1}{3}} \int_{h(u^2)}^{u^1} \rho^{-\frac{2}{3}}(u^1, u^2)du^1\right\} \psi_s(u^1, u^2, \gamma) \tag{3, 4}$$

Again we assume that each ψ_s can be expanded as an asymptotic series of the form,

$$\psi_s = k^{r_1} \sum_{n=0}^{\infty} k^{-\frac{n}{3}} \psi_{ns}$$

If we put this in equation (2, 9) and equate the coefficient of the highest power of k to zero we get the ordinary differential equation for ψ_{os}

$$\frac{1}{\gamma^2}\frac{\partial^2\psi_{os}}{\partial\gamma^2} + \frac{\partial\psi_{os}}{\partial\gamma}\left\{-\frac{1}{\gamma^3} - \frac{2it_s}{\gamma^2} + 2i\right\} + \psi_{os}\left\{\frac{it_s}{\gamma^3} - \frac{t_s^2}{\gamma^2} + \frac{i}{\gamma}\right\} = 0$$

with the solution

$$\psi_{os} = \exp\left\{i(t_s\gamma - \tfrac{1}{3}\gamma^3)\right\} W_1(t_s-p)\ H(u^1, u^2, s)$$

where $p = 2^{-\frac{2}{3}}\gamma$

The asymptotic solution we found has the form

$$\widetilde{\varphi}_d \approx k^{r_1} \sum_{s=0}^{\infty} \exp\left\{ik(S_0+u^3)+it_s\left(\frac{k}{2}\right)^{\frac{1}{3}} \int_{h(u^2)}^{u^1} \rho^{-\frac{2}{3}}(u^1, u^2)du^1 + i(t_s\gamma-\tfrac{1}{3}\gamma^3)\right\} \times$$

$$\times W_1(t_s-p)\ H(u^1, u^2, s)$$

We continue this solution at finite distance from the object and match it with (3,3). This matching leads to $H(u^1,u^2,s)$ and r_1. After some calculations we find at finite distance from the object:

$$\tilde{\varphi}_d \approx \frac{\left(\frac{k}{2}\right)^{-\frac{1}{6}} \sqrt{\pi}\ \rho^{\frac{1}{3}}(u^1,u^2)\, g^{\frac{1}{4}}}{\left[(u^3-u^1)\left\{g^{\frac{1}{2}}+(u^3-u^1)\ \frac{\partial g^{\frac{1}{2}}}{\partial u^1}\right\}\right]^{\frac{1}{2}}} \exp\left\{ik(S_o+u^3)+\frac{\pi i}{4}\right\}$$

$$\sum_{s=0}^{\infty} \frac{\exp\left\{it_s\left(\frac{k}{2}\right)^{\frac{1}{3}} \int_{h(u^2)}^{u^1} \rho^{-\frac{2}{3}}(u^1,u^2)du^1\right\}}{\left\{W_1^{1}(t_s)\right\}^2} \tag{3,6}$$

and near the object

$$\tilde{\varphi}_d \approx \frac{\sqrt{\pi}\ g^{\frac{1}{4}} \exp\left\{ik(S_o+u^3)-i\frac{2}{3}p^{\frac{3}{2}}\right\}}{\left\{g^{\frac{1}{2}}+(u^3-u^1)\ \frac{\partial g^{\frac{1}{2}}}{\partial u^1}\right\}^{\frac{1}{2}}} \sum_{s=0}^{\infty} \frac{e^{it_s q}\, W_1(t_s-p)}{\left\{W_1^{1}(t_s)\right\}^2} \tag{3,7}$$

with

$$p = \left(\frac{k}{2}\right)^{\frac{2}{3}} \rho^{-\frac{4}{3}}(u^3-u^1)^2$$

$$q = \left(\frac{k}{2}\right)^{\frac{1}{3}} \left\{\int_{h(u^2)}^{u^1} \rho^{-\frac{2}{3}}(u^1,u^2)du^1 + \rho^{-\frac{2}{3}}(u^3-u^1)\right\}$$

All these solutions are singular in the points

$$g^{\frac{1}{2}} + (u^3-u^1)\ \frac{\partial g^{\frac{1}{2}}}{\partial u^1} = 0$$

Hence all the derived solutions are valid apart from these singularities, which we call caustic points and therefore the assumption that the gradients of $\tilde{\varphi}$ are moderate is violated and therefore the term with the highest power of k is not the leading one any more. It is possible to construct an asymptotic solution near the caustic with the help of our method. In this article we consider only points at finite distance from these caustics.

We are now able to give the complete solution in the shadow region at finite distance from the caustics. We must sum up all ray contributions in a point. This is a geometrical problem which is not solved here.

4. *The incident wave is spherical.*

Remembering that the geometry which we consider depends on the incident wave it is obvious though the object is the same as the one of section 3 the coordinate system which we introduce here is a different one. However we use the same notation.

We now take as a solution of (2,9) the form:

$$\chi = A \int_C e^{it(q-q_0)}\ W_1(t-p_o)\left\{v(t-p)-\frac{v(t)}{W_1(t)}\ W_1(t-p)\right\}dt$$

where p_o, q_o are the p, q coordinates of the source. The shadow boundary on the object is called $u^1 = h(u^2)$. And as we mentioned in section 3 the radius of curvature of the wave fronts of the incoming waves on the shadow boundary is

$$\rho_1 = \frac{g^{\frac{1}{2}}}{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}} \tag{4,1}$$

And in this case ρ_1 is the distance from a point on the shadow boundary $\{h(u^2), u^2\}$ to the source. We take the origin of u^3 in the source, hence $S_c = 0$. Along the diffracted ray the arclength is $u^3 - u^1$. We can derive the function $u^1 = h(u^2)$ eliminating u^1 from

$$u^1 = \frac{g^{\frac{1}{2}}}{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}}$$

As a solution in the shadow region, near the shadow boundary, we have

$$\tilde{\varphi}_d \approx \frac{A\ k^{r_2}\ \exp\left\{iku^3 - i\frac{2}{3}(p^{\frac{3}{2}} + p_o^{\frac{3}{2}})\right\}}{\left\{\frac{g^{\frac{1}{2}}}{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}} + (u^3 - u^1)\right\}^{\frac{1}{2}}} \times$$

$$\times \int_C e^{it(q-q_o)}\ W_1(t-p_o)\left\{v(t-p) - \frac{v(t)}{W_1(t)}\ W_1(t-p)\right\}dt \tag{4,2}$$

This solution should be equal to $\tilde{\varphi}_{inc}$ on the shadow boundary at finite distance from the object. Hence we expand (4,2) for large values of p_o and p and find

$$\tilde{\varphi}_d = 2A\sqrt{\pi}\,k^{r_2}\ \rho^{\frac{1}{3}}\left\{h(u^2), u^2\right\}\left(\frac{k}{2}\right)^{-\frac{1}{6}}\frac{\exp(iku^3 + \frac{\pi i}{4})}{u^3} =$$

$$= \tilde{\varphi}_{inc} = \frac{\exp(iku^3)}{u^3}$$

from which follows

$$A = \frac{2^{-\frac{1}{6}}e^{-\frac{1}{4}\pi i}}{2\sqrt{\pi}}\ \rho^{-\frac{1}{3}}\left\{h(u^2), u^2\right\} \quad , \quad r_2 = \frac{1}{6}$$

Considering the case that the source is at finite distance from the object and the observation point near the object, we expand the solution for large values of p_o. Because the observation point may be at finite distance from the shadow boundary we use the variables of (3,7) and get the solution

$$\tilde{\varphi} \approx \frac{\exp(iku^3 - i\frac{2}{3}p^{\frac{3}{2}})}{2\left[\frac{\pi g^{\frac{1}{2}}}{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}}\left(\frac{g^{\frac{1}{2}}}{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}} + (u^3 - u^1)\right)\right]^{\frac{1}{2}}}\int_C e^{itq}\left\{v(t-p) - \frac{v(t)}{W_1(t)}W_1(t-p)\right\}dt \tag{4,3}$$

with

$$p = \left(\frac{k}{2}\right)^{\frac{2}{3}} \rho^{-\frac{4}{3}} \left(u^3 - u^1\right)^2$$

$$q = \left(\frac{k}{2}\right)^{\frac{1}{3}} \left\{ \int_{h(u^2)}^{u^1} \rho^{-\frac{2}{3}}(u^1, u^2) du^1 + \rho^{-\frac{2}{3}}(u^3 - u^1) \right\}$$

This solution can be expressed as a sum of residues in the region where this sum is rapidly convergent, we get

$$\tilde{\varphi}_d \approx \frac{-\pi i \exp\left\{iku^3 - i\frac{2}{3}p^{\frac{3}{2}}\right\}}{\left\{\frac{\pi g^{\frac{1}{2}}}{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}}\left(\frac{g^{\frac{1}{2}}}{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}} + (u^3 - u^1)\right)\right\}^{\frac{1}{2}}} \sum_{s=0}^{\infty} e^{itsq} \frac{W_1(ts - p)}{\left\{W_1^1(t_s)\right\}^2} \tag{4, 4}$$

bearing in mind that $W_1(t_s) = 0$

A solution at finite distance from the object will be gained by expanding $W_1(t_s - p)$ for large values of p.

$$\tilde{\varphi}_d \approx \frac{\left(\frac{k}{2}\right)^{-\frac{1}{6}} \sqrt{\pi}\, \rho^{\frac{1}{3}}(u^1, u^2) \exp\left\{iku^3 - \frac{\pi i}{4}\right\}}{\left\{\frac{g^{\frac{1}{2}}}{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}} (u^3 - u^1) \left(\frac{g^{\frac{1}{2}}}{\frac{\partial g^{\frac{1}{2}}}{\partial u^1}} + (u^3 - u^1)\right)\right\}} \times$$

$$\times \frac{\sum_{s=0}^{\infty} \exp\left\{its\left(\frac{k}{2}\right)^{\frac{1}{3}} \int_{h(u^2)}^{u^1} \rho^{-\frac{2}{3}}(u^1, u^2) du^1\right\}}{\left\{W_1^1(t_s)\right\}^2} \tag{4, 5}$$

Again the final solution is a superposition of all ray contributions in the point of observation.

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